

EIGENVALUE ESTIMATES FOR A CLASS OF ELLIPTIC DIFFERENTIAL OPERATORS IN DIVERGENCE FORM

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ABSTRACT. In this paper we compute estimates for eigenvalues of a class of linear second-order elliptic differential operators in divergence form. In all the cases, we assume the Dirichlet boundary condition on a domain in a complete Riemannian manifold. Our estimates are based upon the Weyl's asymptotic formula. In particular, for the drifting Laplacian we derive three inequalities that extend results of Cheng and Yang obtained for a domain in the unit sphere and for a domain in the projective space. Moreover, we find a generalisation of the lower bound for the mean of the first k eigenvalues based on the Pólya's conjecture. Finally, we obtain an estimate that give us an upper bound for the k -th eigenvalue of the drifting Laplacian operator.

1. INTRODUCTION

The Dirichlet problem for the Laplacian operator in a domain $\Omega \subset \mathbb{R}^2$ is the following

$$\begin{cases} -\Delta u &= \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

The constant λ that satisfies (1.1) is the eigenvalue and the corresponding solution $u \in \mathcal{C}^2(\Omega)$ is the eigenfunction. The existence of solutions for (1.1) is known as the “membrane problem”. A fixed membrane can be thought of as the skin of a drum. As we drum, the skin vibrates with frequencies which are precisely the eigenvalues of the Laplacian. It is also important to notice that the eigenvalues depend on the shape, material of the membrane, etc.

Eigenvalues of differential operators find their applications in many areas. For instance, in quantum mechanics quantities like energy, momentum and position are represented by hermitian operators in a Hilbert space. The eigenvalues of the operator that corresponds to the energy are interpreted as the possible values of energy that the system can attain. In addition, the gap between them is simply the gap between the energy levels.

In 1911, Weyl studied problem (1.1) being the first to publish a proof of the asymptotic behavior of the eigenvalues of the Laplacian on domains in \mathbb{R}^n [21]. However, Rayleigh had already studied such behavior few years earlier which had actually been conjectured by Sommerfeld and Lorentz. The Weyl's asymptotic formula is given by

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \text{vol}(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} \quad (k \rightarrow \infty) \quad (1.2)$$

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where ω_n is the volume of the unit ball in \mathbb{R}^n . The discovery of Weyl has become an important tool for comparison of estimates related to eigenvalues, not only for the Laplacian operator, but also for the more general elliptic operators. Furthermore, the domains in \mathbb{R}^n could be replaced by domains in Riemannian manifolds.

We note that from the Weyl's asymptotic formula we can obtain

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}} \quad (1.3)$$

and

$$\frac{1}{k} \sum_{i=1}^k \lambda_i^2 \sim \frac{n}{n+4} \frac{16\pi^4}{(\omega_n \text{vol} \Omega)^{\frac{4}{n}}} k^{\frac{4}{n}}. \quad (1.4)$$

Good estimates are those that provide the best bounds taking into account the relations (1.2), (1.3) and (1.4).

These latter relations brought about the universal inequalities for the Laplacian operator in domains of \mathbb{R}^n . One example is the Payne-Pólya-Weinberger inequality [16, 17]. Payne, Pólya and Weinberger introduced the test-function method and proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^k \lambda_i. \quad (1.5)$$

Hile and Protter [11] improved inequality (1.5) as follows

$$\sum_{i=1}^k \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}.$$

Further, Yang [23] obtained the *Yang's first inequality*

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i$$

and *Yang's second inequality*

$$\lambda_{k+1} \leq \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^k \lambda_i.$$

Moreover, he obtained

$$\lambda_{k+1} - \lambda_k \leq 2 \left[\left(\frac{2}{n} \frac{1}{k} \sum_{i=1}^k \lambda_i \right)^2 - \left(1 + \frac{4}{n} \right) \frac{1}{k} \sum_{i=1}^k \left(\lambda_i - \frac{1}{k} \sum_{j=1}^k \lambda_j \right)^2 \right]^{\frac{1}{2}}. \quad (1.6)$$

The Weyl's asymptotic formula ensures that the estimate obtained by Yang is better than the one of Payne-Pólya-Weinberger.

An alternative technique to estimate the eigenvalues of the Laplacian on domains in \mathbb{R}^n is the recursion formula of Cheng-Yang [7]. Using this formula, Cheng and Yang obtained the following result for a domain Ω in \mathbb{R}^n

$$\lambda_{k+1} \leq \mathbf{C}(n, k) k^{\frac{2}{n}} \lambda_1, \quad (1.7)$$

where $\mathbf{C}(n, k)$ is a constant. Again, the Weyl's asymptotic formula ensures that this bound is the best possible with respect to $k^{\frac{2}{n}}$.

Another inequality involving the eigenvalues of Laplacian for the Dirichlet problem was conjectured by Pólya. In a domain $\Omega \subset \mathbb{R}^n$ he proposed the inequality

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}$$

for $k \in \{1, 2, \dots\}$. This inequality is known as the *Pólya's conjecture*. In relation to this conjecture, Li and Yau [14] proved to be valid the inequality

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}. \quad (1.8)$$

Let us now consider the Laplacian operator on a compact Riemannian manifold M with boundary. Cheng and Yang [6] studied the Dirichlet problem in this case and proved an inequality of a more general form. They obtained

$$\sum_{i=1}^k (\Lambda_i^1)^2 \int_M u_i^2 |\nabla h|^2 dM \leq \sum_{i=1}^k \Lambda_i^1 \int_M (2\langle \nabla h, \nabla u_i \rangle + u_i \Delta h)^2 dM \quad (1.9)$$

where $\Lambda_i^1 = \lambda_{k+1} - \lambda_i$, $h \in \mathcal{C}^3(M) \cap \mathcal{C}^2(\partial M)$ and u_i is an eigenfunction of the eigenvalue λ_i . Applying (1.9) in a domain Ω of a complete Riemannian manifold M^n and using Nash's Theorem [15], Chen and Cheng [4] obtained the following generalisation of the Yang's first inequality

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left(\lambda_i + \frac{n^2}{4} H_0^2 \right) \quad (1.10)$$

and a generalisation of the Yang's second inequality

$$\lambda_{k+1} + \frac{n^2}{4} H_0^2 \leq \left(1 + \frac{4}{n} \right) \frac{1}{k} \sum_{i=1}^k \left(\lambda_i + \frac{n^2}{4} H_0^2 \right),$$

where $H_0 = \sup_{\bar{\Omega}} \|H\|$ and $\|H\|$ is the norm of the mean curvature vector of the immersion of M^n into \mathbb{R}^m . Furthermore, using the recursion formula of Cheng-Yang, they obtained

$$\lambda_{k+1} + \frac{n^2}{4} H_0^2 \leq \bar{\mathbf{C}}(n, k) k^{\frac{2}{n}} \left(\lambda_1 + \frac{n^2}{4} H_0^2 \right),$$

where $\bar{\mathbf{C}}(n, k)$ is constant.

Cheng and Yang [8] employed the recursion formula and found a relation for the first k eigenvalues of the Laplacian operator defined in a bounded domain of a n -dimensional Riemannian manifold isometrically immersed in a Euclidean space. Namely,

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}} \quad \text{for } k \geq 1. \quad (1.11)$$

Let $Lu = \Delta u - \langle \nabla u, \nabla \eta \rangle$ be the drifting Laplacian operator (also called η -Laplacian, since $Lu = e^\eta \text{div}(e^{-\eta} \nabla u)$). Xia and Xu [22] generalised (1.9) for the drifting Laplacian as

$$\sum_{i=1}^k (\Lambda_i^1)^2 \int_M u_i^2 |\nabla h|^2 dm \leq \sum_{i=1}^k \Lambda_i^1 \int_M (2\langle \nabla h, \nabla u_i \rangle + u_i Lh)^2 dm \quad (1.12)$$

where $\Lambda_i^1 = \lambda_{k+1} - \lambda_i$ and $dm = e^{-\eta} dM$. They also considered a domain Ω of a complete Riemannian manifold M isometrically immersed in a Euclidean space to prove that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (4\lambda_i + 4\eta_0 \sqrt{\lambda_i} + n^2 H_0^2 + \eta_0^2), \quad (1.13)$$

where $\eta_0 = \max_{\overline{\Omega}} |\nabla \eta|$ and $H_0 = \sup_{\overline{\Omega}} \|H\|$.

In this paper we consider Ω to be a domain of a complete Riemannian manifold M and T a symmetric positive definite $(1,1)$ -tensor on M in order to study the eigenvalue problem with Dirichlet condition on boundary

$$\begin{cases} -\mathcal{L}u &= \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\mathcal{L}u := \operatorname{div}_{\eta}(T(\nabla u)) = \operatorname{div}(T(\nabla u)) - \langle \nabla \eta, T(\nabla u) \rangle. \quad (1.14)$$

The main results of this paper are the following.

Theorem 1. *Let Ω be a domain in an n -dimensional complete Riemannian manifold M isometrically immersed in \mathbb{R}^m , λ_i be the i -th eigenvalue of \mathcal{L} with eigenfunction u_i . Then*

$$\operatorname{tr}(T) \sum_{i=1}^k (\Lambda_i^1)^2 \leq \sum_{i=1}^k \Lambda_i^1 \left((m-n)^2 A_0^2 T_*^2 + (T_0 + T_* \eta_0)^2 + 4(T_0 + T_* \eta_0) \lambda_i^{\frac{1}{2}} + 4\lambda_i \right),$$

where $\Lambda_i^1 = \lambda_{k+1} - \lambda_i$, $A_0 = \max\{\sup_{\overline{\Omega}} |A_{e_k}|, k = n+1, \dots, m\}$, A_{e_k} is the Weingarten operator of the immersion in the direction e_k , $\eta_0 = \sup_{\overline{\Omega}} |\nabla \eta|$, $T_* = \sup_{\overline{\Omega}} |T|$ and $T_0 = \sup_{\overline{\Omega}} |\operatorname{tr}(\nabla T)|$.

Theorem 2. *Let Ω be a domain in an n -dimensional complete Riemannian manifold M isometrically immersed in \mathbb{R}^m with mean curvature H and λ_i be the i -th eigenvalue of the drifting Laplacian operator. Then*

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left[\lambda_i + \frac{n^2 H_0^2 + \eta_0^2 + 2\bar{\eta}_0}{4} \right], \quad (1.15)$$

where $\eta_0 = \sup_{\overline{\Omega}} |\nabla \eta|$, $\bar{\eta}_0 = \sup_{\overline{\Omega}} \mathcal{L}\eta$ and $H_0 = \sup_{\overline{\Omega}} \|H\|$ are constants.

By setting $v_i := \lambda_i + \frac{n^2 H_0^2 + \eta_0^2 + 2\bar{\eta}_0}{4}$ we readily observe that inequality (1.15) is equivalent to

$$\sum_{i=1}^k (v_{k+1} - v_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (v_{k+1} - v_i) v_i, \quad (1.16)$$

which has exactly the same form as the first Yang's inequality. Bearing in mind this observation we obtain the following corollaries of Theorem 2. Firstly, we derive three inequalities that extend the results of Cheng and Yang obtained for a domain in the unit sphere $\mathbb{S}^n(1)$ and for a domain in the projective space $\mathbb{CP}^n(4)$ [5, 6]. Secondly, we find a generalisation of the lower bound for the mean of the first k eigenvalues based on the Pólya's conjecture. Finally, we extend the estimate (1.7) that give us an upper bound for the k -th eigenvalue of the drifting Laplacian operator.

2. MOTIVATING THE DEFINITION OF THE OPERATOR \mathcal{L}

In this section we will establish the tools necessary to work with the operator defined in (1.14) which enable us to obtain much more general results. We believe that this operator would also be useful in obtaining rigidity results or characterizing some known geometric objects.

Given a Riemannian manifold, to each $X \in \mathfrak{X}(M)$ we associate the $(0,1)$ -tensor $X^\flat : \mathfrak{X}(M) \rightarrow C^\infty(M)$, given by

$$X^\flat(Y) = \langle X, Y \rangle, \quad \text{for all } Y \in \mathfrak{X}(M).$$

It will also be convenient to consider the musical isomorphism $^\sharp : \mathfrak{X}^*(M) \rightarrow \mathfrak{X}(M)$, that is, the inverse of the canonical mapping $^\flat : \mathfrak{X}(M) \rightarrow \mathfrak{X}^*(M)$.

In this paper we will be constantly using the identification of a $(0,2)$ -tensor T with its associated $(1,1)$ -tensor by the equation

$$\langle T(X), Y \rangle = T(X, Y).$$

In particular, the metric tensor \langle, \rangle will be identified with the $(1,1)$ -tensor of the identity I in $\mathfrak{X}(M)$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis in $T_p M$ and S be a $(1,1)$ -tensor with adjoint S^* . Recall that the *Hilbert-Schmidt inner product* is given by

$$\langle T, S \rangle := \text{tr}(TS^*) = \sum_i \langle TS^*(e_i), e_i \rangle = \sum_i \langle S^*(e_i), T^*(e_i) \rangle = \sum_i \langle T(e_i), S(e_i) \rangle.$$

The divergence of a $(1,1)$ -tensor T in (M, \langle, \rangle) is defined as the $(0,1)$ -tensor

$$(\text{div} T)(v)(p) = \text{tr}(w \mapsto (\nabla_w T)(v)(p)),$$

where $p \in M$, $v, w \in T_p M$, ∇ stands for the covariant derivative of T and tr is the trace calculated in the metric \langle, \rangle .

Furthermore, we note that the Riemannian metric on M induces the metric

$$\langle X^\flat, Y^\flat \rangle = \langle X, Y \rangle,$$

where $X, Y \in T_p M$ and $X^\flat, Y^\flat \in T_p^* M$. We easily compute that

$$\begin{aligned} \langle \text{div} T, Z^\flat \rangle &= \langle (\text{div} T)^\sharp, Z \rangle = \sum_i \langle (\text{div} T)^\sharp(e_i), Z(e_i) \rangle = \sum_i (\text{div} T)(e_i) \langle Z, e_i \rangle \\ &= (\text{div} T) \left(\sum_i \langle Z, e_i \rangle e_i \right) = (\text{div} T)(Z). \end{aligned}$$

When there is no danger of confusion we shall omit the symbol “ $^\sharp$ ” for the sake of simplicity.

For any vector field $X \in \mathfrak{X}(M)$ we can consider the $(1,1)$ -tensor ∇X given by $\nabla X(Y) = \nabla_Y X$, for all $Y \in \mathfrak{X}(M)$. In this way, the divergence of X is given by $\text{div} X = \text{tr}(\nabla X)$. We will fix a function $\eta \in C^\infty(M)$, to define the η -divergence of X as follows

$$\text{div}_\eta X := \text{div} X - \langle \nabla \eta, X \rangle.$$

It is immediate from the usual properties of divergence of vector fields that, for all $f \in C^\infty(M)$ we have

$$\begin{aligned} \text{div}_\eta(X + Y) &= \text{div}_\eta X + \text{div}_\eta Y, \\ \text{div}_\eta(fX) &= f \text{div}_\eta X + \langle \nabla f, X \rangle, \\ \text{div}(e^{-\eta} X) &= e^{-\eta} \text{div}_\eta X. \end{aligned}$$

Suppose that (M, \langle, \rangle) is an oriented Riemannian manifold, and let dM denote its Riemannian volume form. We use the weighted measures $dm = e^{-\eta}dM$ and $d\mu = e^{-\eta}d\partial M$. If ν is the outward normal vector field on the boundary ∂M and X is a tangent vector field with compact support on M , then

$$\int_M \operatorname{div}_\eta X dm = \int_M e^{-\eta} \operatorname{div}_\eta X dM = \int_M \operatorname{div}(e^{-\eta} X) dM = \int_{\partial M} \langle X, \nu \rangle d\mu$$

which is the expression of the divergence theorem (or Stokes theorem) for the *weighted manifolds* (M, dm) . Note that the η -Laplacian operator L is given by

$$Lf = \operatorname{div}_\eta \nabla f$$

for all $f \in C^\infty(M)$. It is immediate that L satisfies analogous properties to those of the Laplacian. For instance, for $f, \ell \in C^\infty(M)$ we have

$$L(f\ell) = fL\ell + \ell Lf + 2\langle \nabla f, \nabla \ell \rangle$$

and an extension of the well-known Bochner formula

$$\frac{1}{2}L|\nabla f|^2 = Ric_\eta(\nabla f, \nabla f) + |\nabla^2 f|^2 + \langle \nabla Lf, \nabla f \rangle,$$

where $Ric_\eta := Ric + \nabla^2 \eta$ is the Bakry-Emery-Ricci tensor. This tensor have been especially studied in the theory of solitons and almost Ricci solitons, since a gradient Ricci soliton $(M, \langle, \rangle, \nabla \eta, \lambda)$ is characterized by $Ric_\eta = \lambda \langle, \rangle$ for some constant λ , while in the case of almost gradient Ricci soliton, λ is a real smooth function in M .

Now let us consider a operator which is an extension of η -Laplacian.

Definition 1. Let T be a symmetric positive definite $(1, 1)$ -tensor in (M, \langle, \rangle) and $\eta \in C^\infty(M)$. We define the (η, T) -divergence operator by

$$\mathcal{L}f := \operatorname{div}_\eta(T(\nabla f)) = \operatorname{div}(T(\nabla f)) - \langle \nabla \eta, T(\nabla f) \rangle$$

for all $f \in C^\infty(M)$.

It is immediate from the properties of div_η and the symmetry of T that

$$\mathcal{L}(f\ell) = f\mathcal{L}\ell + \ell\mathcal{L}f + 2T(\nabla f, \nabla \ell)$$

for $f, \ell \in C^\infty(M)$. Moreover, we define the η -divergence of T by

$$\operatorname{div}_\eta T := \operatorname{div} T - d\eta \circ T.$$

Alencar, Neto and Zhou [1] have recently proved a new Bochner-type formula and applied it to the operator which was introduced by Cheng and Yau [9], namely

$$\square f = \operatorname{tr}(\nabla^2 f \circ T) = \langle \nabla^2 f, T \rangle,$$

where $f \in C^\infty(M)$ and T is a symmetric $(1, 1)$ -tensor. Such formula is given by

$$\frac{1}{2}\square(|\nabla f|^2) = \langle \nabla(\square f), \nabla f \rangle + R_T(\nabla f, \nabla f) + \langle \nabla^2 f, \nabla^2 f \circ T \rangle - \langle \nabla^2 f, \nabla_{\nabla f} T \rangle \quad (2.1)$$

where $R_T(X, Y) = \operatorname{tr}(T \circ (Z \rightarrow R(X, Z)Y))$ and R is the curvature tensor of M .

A straightforward computation gives us

$$\operatorname{div}_\eta(T(f\nabla h)) = f\langle \operatorname{div}_\eta T, \nabla h \rangle + f\langle \nabla^2 h, T \rangle + T(\nabla f, \nabla h) \quad (2.2)$$

where T is symmetric $(0, 2)$ -tensor and η, f, h are real smooth functions on M .

From (2.2) we get the next equation

$$\mathcal{L}f = \langle \operatorname{div}_\eta T, \nabla f \rangle + \square f. \quad (2.3)$$

We point out that equation (2.3) relates the operators \mathcal{L} , $\operatorname{div}_\eta T$ and \square . It also gives us a *Bochner-type formula for the operator \mathcal{L}* on Riemannian manifold, namely:

$$\frac{1}{2}\mathcal{L}(|\nabla f|^2) = \langle \nabla(\mathcal{L}f), \nabla f \rangle + R_{\eta,T}(\nabla f, \nabla f) + \langle \nabla^2 f, \nabla^2 f \circ T \rangle - \langle \nabla^2 f, \nabla_{\nabla f} T \rangle$$

where $R_{\eta,T} := R_T - \nabla(\operatorname{div}_\eta T)^\sharp$.

Indeed, from equation (2.3) we can deduce

$$\begin{aligned} \langle \nabla(\mathcal{L}f), \nabla f \rangle &= \langle \nabla_{\nabla f} \operatorname{div}_\eta T, \nabla f \rangle + \langle \operatorname{div}_\eta T, \nabla_{\nabla f} \nabla f \rangle + \langle \nabla(\square f), \nabla f \rangle \\ &= \nabla(\operatorname{div}_\eta T)^\sharp(\nabla f, \nabla f) + \frac{1}{2}\langle \operatorname{div}_\eta T, \nabla(|\nabla f|^2) \rangle + \langle \nabla(\square f), \nabla f \rangle \\ &= \nabla(\operatorname{div}_\eta T)^\sharp(\nabla f, \nabla f) + \frac{1}{2}\mathcal{L}(|\nabla f|^2) - \frac{1}{2}\square(|\nabla f|^2) + \langle \nabla(\square f), \nabla f \rangle, \end{aligned}$$

that is,

$$\frac{1}{2}\mathcal{L}(|\nabla f|^2) = \langle \nabla(\mathcal{L}f), \nabla f \rangle - \nabla(\operatorname{div}_\eta T)^\sharp(\nabla f, \nabla f) + \frac{1}{2}\square(|\nabla f|^2) - \langle \nabla(\square f), \nabla f \rangle.$$

Therefore, our assertion follows from Bochner-type formula (2.1).

Having proven the Bochner-type formula for \mathcal{L} , it is now most natural to study hypersurfaces with constant mean curvature k by means of the properties of the operator \mathcal{L} as well as to obtain some generalizations in the spirit of Lichnerowicz's and Obata's theorems. Using the same technique one can even hope to generalize the recent results obtained by Alencar, Neto and Zhou [1]. Furthermore, we can also obtain some generalizations of the work of Uhlenbeck [20], Berger [3] and Soulf and Ilias [19].

For orientable compact Riemannian manifolds, Cheng and Yau proved that the operator \square is self-adjoint if and only if $\operatorname{div} T = 0$. An alternative proof of this fact follows from equation (2.2). Indeed, from this equation and the divergence theorem, we get

$$\int_M (f\square h - h\square f) dM = \int_M (h\langle \operatorname{div} T, \nabla f \rangle - f\langle \operatorname{div} T, \nabla h \rangle) dM$$

for all $f, h \in C^\infty(M)$. Hence, the operator \square is self-adjoint if and only if $\operatorname{div} T = 0$. In particular, equation (2.3) reduces to

$$\mathcal{L}f = -T(\nabla \eta, \nabla f) + \square f.$$

For instance, denoting $R = \operatorname{tr}(Ric)$, it is well-known that $\operatorname{div} Ric = \frac{dR}{2}$ and $\operatorname{div}(RI) = dR$, so the tensor $G := Ric - \frac{R}{2}\langle \cdot, \cdot \rangle$ has null divergence, and therefore $\square f = \langle \nabla^2 f, G \rangle$ is self-adjoint. Then,

$$\mathcal{L}f = -G(\nabla \eta, \nabla f) + \langle \nabla^2 f, G \rangle.$$

3. THE DIRICHLET PROBLEM

In what follows $(M, \langle \cdot, \cdot \rangle)$ is a complete Riemannian manifold and the domain $\Omega \subset M$ is connected, bounded and open. We observe that the divergence theorem remains true in the form $\int_\Omega \mathcal{L}f d\mu = \int_{\partial\Omega} T(\nabla f, \nu) d\mu$, where $d\mu = e^{-\eta} d\partial\Omega$ is the weight volume form on the boundary $\partial\Omega$ induced by the outward normal vector ν on $\partial\Omega$. Thus, the integrate by parts formula is

$$\int_\Omega \ell \mathcal{L}f d\mu = - \int_\Omega T(\nabla \ell, \nabla f) d\mu + \int_{\partial\Omega} \ell T(\nabla f, \nu) d\mu$$

for all $f, \ell \in \mathcal{C}^\infty(\Omega)$. Therefore, \mathcal{L} is self-adjoint in the space of all functions in $L^2(\Omega, \text{dm})$ that vanish in $\partial\Omega$. Thus the eigenvalue problem

$$\begin{cases} -\mathcal{L}u &= \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

has a real and discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow \infty$, where each λ_i is repeated according to its multiplicity. In particular, for an eigenfunctions u_i we have

$$\lambda_i = - \int_{\Omega} u_i \mathcal{L} u_i \text{dm} = \int_{\Omega} T(\nabla u_i, \nabla u_i) \text{dm}.$$

Operators in divergence form have already been studied in different settings [2, 13, 18]. For instance, for $T = I$ the operator \mathcal{L} is the η -Laplacian, it is closely related to eigenvalues problems in spaces with warped product [10, 12]. For example, when $\Omega \subset \mathbb{R}^n$, we consider the metric $g = g_0 + e^{-\eta} d\theta^2$ on the product $\Omega \times \mathbb{S}^1$, where g_0 stands for the Riemannian metric in the domain Ω , while $d\theta^2$ is the canonical metric of \mathbb{S}^1 , then the scalar curvature of the metric g is given by $f = \frac{1}{4}(2\Delta\eta - |\nabla\eta|^2)$. We mention that the modified scalar curvature of a metric g and a dilatation function η , as introduced by Perelman, is $R^m = R + 4f$, where R is the scalar curvature of g .

The Weyl's asymptotic expansion (1.2) and its immediate consequences (1.3) and (1.4) will be of paramount importance to obtaining our estimates. An immediate implication of (1.3) is that

$$\frac{4}{nk} \sum_{i=1}^k \lambda_i \sim \frac{4}{n+2} \frac{4\pi^2}{(\omega_n \text{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}$$

and from (1.5) we obtain

$$\lambda_{k+1} - \lambda_k \sim \frac{4}{n+2} \frac{4\pi^2}{(\omega_n \text{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

Analogous reasoning allows us to deduce from (1.3), (1.4) and (1.6) that

$$\lambda_{k+1} - \lambda_k \leq 2 \left[\left(\frac{2}{n} \frac{1}{k} \sum_{i=1}^k \lambda_i \right)^2 - \left(1 + \frac{4}{n} \right) \frac{1}{k} \left(\sum_{i=1}^k \lambda_i^2 - \frac{1}{k} \left(\sum_{j=1}^k \lambda_j \right)^2 \right) \right]^{\frac{1}{2}} \sim 0.$$

Therefore, the estimate (1.6) is better than (1.5).

In Section 4 we will make extensions of (1.7) and (1.8) for the particular case when \mathcal{L} is the η -Laplacian.

Now we start with the proofs of principal results. Firstly, we will demonstrate the proposition below which extends (1.9) and (1.12) for \mathcal{L} . For this, we assume that the eigenfunctions associated to the problem (3.1) are orthonormal in $L^2(\Omega, \text{dm})$.

Proposition 1. *Let λ_i be the i -th eigenvalue of Problem (3.1) and u_i its corresponding eigenfunction. Then, for $h \in \mathcal{C}^3(\Omega) \cap \mathcal{C}^2(\partial\Omega)$, and k integer, is valid*

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} u_i^2 T(\nabla h, \nabla h) \text{dm} \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \int_{\Omega} (u_i \mathcal{L} h + 2T(\nabla h, \nabla u_i))^2 \text{dm}.$$

Proof. By the inequality of Rayleigh-Ritz, we have

$$\lambda_{k+1} \leq -\frac{\int_{\Omega} \psi \mathcal{L} \psi \, \mathrm{d}m}{\int_{\Omega} \psi^2 \, \mathrm{d}m}, \quad (3.2)$$

for any no null function $\psi : \Omega \rightarrow \mathbb{R}$ satisfying

$$\psi|_{\partial\Omega} = 0 \quad \text{and} \quad \int_{\Omega} \psi u_i \, \mathrm{d}m = 0, \quad \forall i = 1, \dots, k.$$

For $h \in \mathcal{C}^3(\Omega) \cap \mathcal{C}^2(\partial\Omega)$ and for each i , we consider

$$\phi_i = hu_i - \sum_{j=1}^k c_{ij} u_j \quad (3.3)$$

where $c_{ij} = \int_{\Omega} hu_i u_j \, \mathrm{d}m$, so that, $\phi_i|_{\partial\Omega} = 0$ and

$$0 = \int_{\Omega} \phi_i u_l \, \mathrm{d}m = \int_{\Omega} hu_i u_l \, \mathrm{d}m - \sum_{j=1}^k c_{ij} \int_{\Omega} u_j u_l \, \mathrm{d}m = \int_{\Omega} hu_i u_l \, \mathrm{d}m - \sum_{j=1}^k c_{ij} \delta_{jl}.$$

Then, we can make $\psi = \phi_i$ in (3.2), i.e.,

$$\lambda_{k+1} \leq -\frac{\int_{\Omega} \phi_i \mathcal{L} \phi_i \, \mathrm{d}m}{\int_{\Omega} \phi_i^2 \, \mathrm{d}m}. \quad (3.4)$$

Since

$$\mathcal{L} \phi_i = \mathcal{L}(hu_i) - \sum_{j=1}^k c_{ij} \mathcal{L} u_j = -\lambda_i hu_i + u_i \mathcal{L} h + 2T(\nabla h, \nabla u_i) + \sum_{j=1}^k c_{ij} \lambda_j u_j,$$

from (3.3) we have

$$\begin{aligned} \phi_i \mathcal{L} \phi_i &= \phi_i (u_i \mathcal{L} h + 2T(\nabla h, \nabla u_i)) - \phi_i \lambda_i hu_i + \sum_{j=1}^k c_{ij} \lambda_j u_j \phi_i \\ &= \phi_i (u_i \mathcal{L} h + 2T(\nabla h, \nabla u_i)) - \lambda_i \phi_i^2 - \sum_{j=1}^k c_{ij} \lambda_i u_j \phi_i + \sum_{j=1}^k c_{ij} \lambda_j u_j \phi_i. \end{aligned} \quad (3.5)$$

Plugging (3.5) in (3.4), we deduce that

$$\lambda_{k+1} \leq -\frac{\int_{\Omega} (\phi_i (u_i \mathcal{L} h + 2T(\nabla h, \nabla u_i)) - \lambda_i \phi_i^2) \, \mathrm{d}m}{\int_{\Omega} \phi_i^2 \, \mathrm{d}m}.$$

Hence

$$(\lambda_{k+1} - \lambda_i) |\phi_i|^2 \leq - \int_{\Omega} \phi_i (u_i \mathcal{L} h + 2T(\nabla h, \nabla u_i)) \, \mathrm{d}m, \quad (3.6)$$

where $|\phi_i|^2 = |\phi_i|_{L^2(\Omega, \mathrm{d}m)}^2$. Now, we will estimate

$$P_i = - \int_{\Omega} \phi_i (u_i \mathcal{L} h + 2T(\nabla h, \nabla u_i)) \, \mathrm{d}m.$$

To do this, we set

$$b_{ij} = - \int_{\Omega} (u_j \mathcal{L} h + 2T(\nabla h, \nabla u_j)) u_i \, \mathrm{d}m.$$

Again we use that $\int_{\Omega} \phi_i u_j \, \mathrm{d}m = 0$ to write

$$\begin{aligned} P_i &= - \int_{\Omega} \phi_i \left(u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i) - \sum_{j=1}^k b_{ij} u_j \right) \mathrm{d}m \\ &\leq |\phi_i| |u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i) - \sum_{j=1}^k b_{ij} u_j|. \end{aligned}$$

Therefore

$$(\lambda_{k+1} - \lambda_i)^2 P_i^2 \leq (\lambda_{k+1} - \lambda_i)^2 |\phi_i|^2 |u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i) - \sum_{j=1}^k b_{ij} u_j|^2. \quad (3.7)$$

From (3.6), $(\lambda_{k+1} - \lambda_i)|\phi_i|^2 \leq P_i$, then (3.7) implies

$$(\lambda_{k+1} - \lambda_i)^2 P_i \leq (\lambda_{k+1} - \lambda_i) |u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i) - \sum_{j=1}^k b_{ij} u_j|^2,$$

whence

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 P_i \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) |u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i) - \sum_{j=1}^k b_{ij} u_j|^2. \quad (3.8)$$

Moreover

$$\begin{aligned} \lambda_i c_{ij} &= - \int_{\Omega} h u_j \mathcal{L} u_i \, \mathrm{d}m = - \int_{\Omega} u_i \mathcal{L} (h u_j) \, \mathrm{d}m \\ &= - \int_{\Omega} u_i (-\lambda_j h u_j + u_j \mathcal{L}h + 2T(\nabla h, \nabla u_j)) \, \mathrm{d}m = \lambda_j c_{ij} + b_{ij}, \end{aligned}$$

thus

$$b_{ij} = (\lambda_i - \lambda_j) c_{ij} \quad \text{and} \quad b_{ij} = -b_{ji}.$$

On the other hand,

$$\begin{aligned} &|u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i) - \sum_{j=1}^k b_{ij} u_j|^2 \\ &= |u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i)|^2 + \sum_{j=1}^k b_{ij}^2 - 2 \sum_{j=1}^k b_{ij} \int_{\Omega} (u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i)) u_j \, \mathrm{d}m \\ &= |u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i)|^2 - \sum_{j=1}^k b_{ij}^2 \\ &= |(u_i \mathcal{L}h + 2T(\nabla h, \nabla u_i))|^2 - \sum_{j=1}^k (\lambda_i - \lambda_j)^2 c_{ij}^2 \end{aligned} \quad (3.9)$$

and

$$\begin{aligned}
P_i &= - \int_{\Omega} (hu_i - \sum_{j=1}^k c_{ij}u_j)(u_i\mathcal{L}h + 2T(\nabla h, \nabla u_i))dm \\
&= - \int_{\Omega} (hu_i^2\mathcal{L}h + 2hu_iT(\nabla h, \nabla u_i))dm + \sum_{j=1}^k c_{ij}b_{ij} \\
&= \int_{\Omega} T(\nabla(hu_i^2), \nabla h)dm - 2 \int_{\Omega} hu_iT(\nabla h, \nabla u_i)dm + \sum_{j=1}^k (\lambda_i - \lambda_j)c_{ij}^2 \\
&= \int_{\Omega} u_i^2T(\nabla h, \nabla h)dm + \sum_{j=1}^k (\lambda_i - \lambda_j)c_{ij}^2. \tag{3.10}
\end{aligned}$$

Substituting (3.9) and (3.10) into (3.8) we obtain

$$\begin{aligned}
&\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} u_i^2T(\nabla h, \nabla h)dm + \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j)c_{ij}^2 \tag{3.11} \\
&\leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \int_{\Omega} (u_i\mathcal{L}h + 2T(\nabla h, \nabla u_i))^2dm - \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2c_{ij}^2.
\end{aligned}$$

We observe that

$$\begin{aligned}
&\sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j)c_{ij}^2 = \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)(\lambda_{k+1} - \lambda_j + \lambda_j - \lambda_i)(\lambda_i - \lambda_j)c_{ij}^2 \\
&= \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)(\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j)c_{ij}^2 + \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)(\lambda_j - \lambda_i)(\lambda_i - \lambda_j)c_{ij}^2 \\
&= - \sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2c_{ij}^2, \tag{3.12}
\end{aligned}$$

since $\sum_{i,j=1}^k (\lambda_{k+1} - \lambda_i)(\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j)c_{ij}^2 = 0$. Finally, the required equation in the proposition follows immediately from (3.11) and (3.12). \square

The next result allow us to extend (1.10) and (1.13) for the operator \mathcal{L} .

Lemma 1. *Let Ω be a domain of an n -dimensional complete Riemannian manifold M isometrically immersed in \mathbb{R}^m , λ_i be the i -th eigenvalue of Problem (3.1) and u_i be the eigenfunction corresponding to λ_i . Let α be the second fundamental form of M and $\alpha \circ T = \alpha(T(\cdot), \cdot)$, then*

$$\begin{aligned}
\text{tr}(T) \sum_{i=1}^k (\Lambda_i^1)^2 &\leq \sum_{i=1}^k \Lambda_i^1 \left[\int_{\Omega} u_i^2 \left(\|\text{tr}(\alpha \circ T)\|^2 + |\text{tr}(\nabla T) - T(\nabla \eta)|^2 \right) dm \right. \\
&\quad \left. + 4 \int_{\Omega} u_i \left(\langle \text{tr}(\nabla T), T(\nabla u_i) \rangle - \langle T(\nabla \eta), T(\nabla u_i) \rangle \right) dm + 4\lambda_i \right],
\end{aligned}$$

where $\Lambda_i^1 = \lambda_{k+1} - \lambda_i$.

Proof. Let $x = (x_1, \dots, x_m)$ be the position vector of the immersion of M in \mathbb{R}^m . Consider $h = x_\ell$ in Proposition 1 and take the sum in ℓ . We have

$$\begin{aligned} & \sum_{i=1}^k (\Lambda_i^1)^2 \int_{\Omega} u_i^2 \sum_{\ell=1}^m T(\nabla x_\ell, \nabla x_\ell) \, \mathrm{d}m \\ & \leq \sum_{i=1}^k \Lambda_i^1 \int_{\Omega} \sum_{\ell=1}^m (u_i \mathcal{L} x_\ell + 2T(\nabla x_\ell, \nabla u_i))^2 \, \mathrm{d}m \\ & = \sum_{i=1}^k \Lambda_i^1 \int_{\Omega} \sum_{\ell=1}^m \left(u_i \operatorname{div}_{\eta}(T(\nabla x_\ell)) + 2T(\nabla x_\ell, \nabla u_i) \right)^2 \, \mathrm{d}m. \end{aligned}$$

So,

$$\begin{aligned} & \sum_{i=1}^k (\Lambda_i^1)^2 \int_{\Omega} u_i^2 \sum_{\ell=1}^m T(\nabla x_\ell, \nabla x_\ell) \, \mathrm{d}m \\ & \leq \sum_{i=1}^k \Lambda_i^1 \int_{\Omega} \sum_{\ell=1}^m \left(u_i^2 (\operatorname{div}_{\eta}(T(\nabla x_\ell)))^2 + 4u_i \operatorname{div}_{\eta}(T(\nabla x_\ell)) T(\nabla x_\ell, \nabla u_i) \right. \\ & \quad \left. + 4T(\nabla x_\ell, \nabla u_i)^2 \right) \, \mathrm{d}m. \end{aligned} \tag{3.13}$$

Denoting the canonical connection of \mathbb{R}^m by ∇^0 and taking $\{e_1, \dots, e_m\}$ a local orthonormal geodesic frame in $p \in M$ adapted to M , we can write

$$\begin{aligned} \nabla^0 x_\ell &= \sum_{i=1}^n e_i(x_\ell) e_i + \sum_{i=n+1}^m e_i(x_\ell) e_i, \\ e_\ell &= \nabla x_\ell + (\nabla x_\ell)^\perp. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\ell=1}^m T(\nabla x_\ell, \nabla u_i)^2 &= \sum_{\ell=1}^m \langle \nabla x_\ell, T(\nabla u_i) \rangle^2 = \sum_{\ell=1}^m \langle e_\ell - (\nabla x_\ell)^\perp, T(\nabla u_i) \rangle^2 \\ &= \sum_{\ell=1}^n \langle e_\ell, T(\nabla u_i) \rangle^2 = |T(\nabla u_i)|^2, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \sum_{\ell=1}^m T(\nabla x_\ell, \nabla x_\ell) &= \sum_{\ell=1}^m \langle \nabla x_\ell, T(\nabla x_\ell) \rangle = \sum_{\ell=1}^m \langle e_\ell - (\nabla x_\ell)^\perp, T(\nabla x_\ell) \rangle \\ &= \sum_{\ell=1}^n \langle e_\ell, T(\nabla x_\ell) \rangle = \sum_{\ell=1}^n \langle T(e_\ell), \nabla x_\ell \rangle \\ &= \sum_{\ell=1}^n \langle e_\ell, T(e_\ell) \rangle = \operatorname{tr}(T). \end{aligned} \tag{3.15}$$

Moreover,

$$\begin{aligned} \operatorname{div}_{\eta}(T(\nabla x)) &:= (\operatorname{div}_{\eta}(T(\nabla x_1)), \dots, \operatorname{div}_{\eta}(T(\nabla x_m))) \\ &= (\operatorname{div}(T(\nabla x_1)) - \langle \nabla \eta, T(\nabla x_1) \rangle, \dots, \operatorname{div}(T(\nabla x_m)) - \langle \nabla \eta, T(\nabla x_m) \rangle) \\ &= \operatorname{div}(T(\nabla x)) - \mathrm{d}\eta \circ T(\nabla x). \end{aligned} \tag{3.16}$$

In this way we have

$$\begin{aligned}
\operatorname{div}(T(\nabla x)) &:= \left(\operatorname{div}(T(\nabla x_1)), \dots, \operatorname{div}(T(\nabla x_m)) \right) \\
&= \left(\sum_{i=1}^n e_i \langle T(\nabla x_1), e_i \rangle, \dots, \sum_{i=1}^n e_i \langle T(\nabla x_m), e_i \rangle \right) \\
&= \sum_{i=1}^n \left(e_i \langle T(\sum_{j=1}^n e_j(x_1) e_j), e_i \rangle, \dots, e_i \langle T(\sum_{j=1}^n e_j(x_m) e_j), e_i \rangle \right) \\
&= \sum_{i,j=1}^n \left(e_i(e_j(x_1) \langle T(e_j), e_i \rangle), \dots, e_i(e_j(x_m) \langle T(e_j), e_i \rangle) \right),
\end{aligned}$$

differentiating we obtain

$$\begin{aligned}
\operatorname{div}(T(\nabla x)) &= \sum_{i,j=1}^n \left(e_i e_j(x_1) \langle T(e_j), e_i \rangle, \dots, e_i e_j(x_m) \langle T(e_j), e_i \rangle \right) \\
&\quad + \sum_{i,j=1}^n \left(e_j(x_1) \langle \nabla_{e_i} T(e_j), e_i \rangle, \dots, e_j(x_m) \langle \nabla_{e_i} T(e_j), e_i \rangle \right) \\
&= \sum_{i,j=1}^n \langle T(e_j), e_i \rangle \left(e_i e_j(x_1), \dots, e_i e_j(x_m) \right) \\
&\quad + \sum_{i,j=1}^n \langle \nabla_{e_i} T(e_j), e_i \rangle \left(e_j(x_1), \dots, e_j(x_m) \right).
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{div}(T(\nabla x)) &= \sum_{i,j=1}^n \langle T(e_j), e_i \rangle (\nabla_{e_i}^0 e_j)(x) + \sum_{i,j=1}^n \langle \nabla_{e_i} T(e_j), e_i \rangle e_j(x) \\
&= \sum_{i,j=1}^n \langle T(e_j), e_i \rangle \nabla_{e_i}^0 e_j + \sum_{i,j=1}^n \langle \nabla_{e_i} T(e_j), e_i \rangle e_j \\
&= \sum_{i,j=1}^n \langle T(e_j), e_i \rangle (\nabla_{e_i} e_j + \alpha(e_i, e_j)) + \sum_{i,j=1}^n \langle \nabla_{e_i} T(e_j), e_i \rangle e_j \\
&= \sum_{i,j=1}^n \langle T(e_j), e_i \rangle \alpha(e_i, e_j) + \sum_{i,j=1}^n \langle \nabla_{e_i} T(e_j), e_i \rangle e_j \\
&= \sum_{j=1}^n \alpha(T(e_j), e_j) + \sum_{i,j=1}^n \langle \nabla_{e_i} T(e_j), e_i \rangle e_j. \tag{3.17}
\end{aligned}$$

Since T is symmetric we have $\langle T(e_j), e_i \rangle = \langle e_j, T(e_i) \rangle$, which implies

$$\langle \nabla_{e_i} T(e_j), e_i \rangle = \langle e_j, \nabla_{e_i} T(e_i) \rangle,$$

and substituting into (3.17) we have

$$\begin{aligned}
\operatorname{div}(T(\nabla x)) &= \sum_{i=1}^n \alpha(T(e_i), e_i) + \sum_{i=1}^n \nabla_{e_i} T(e_i) \\
&= \sum_{i=1}^n \alpha(T(e_i), e_i) + \sum_{i=1}^n (\nabla_{e_i} T)(e_i) \\
&= \sum_{i=1}^n \alpha(T(e_i), e_i) + \sum_{i=1}^n (\nabla T)(e_i, e_i) \\
&= \operatorname{tr}(\alpha(T(\cdot), \cdot)) + \operatorname{tr}(\nabla T),
\end{aligned} \tag{3.18}$$

where

$$\operatorname{tr}(\alpha(T(\cdot), \cdot)) := \sum_{i=1}^n \alpha(T(e_i), e_i) \in \mathfrak{X}(M)^\perp \quad \text{and} \quad \operatorname{tr}(\nabla T) := \sum_{i=1}^n (\nabla T)(e_i, e_i) \in \mathfrak{X}(M).$$

On the other hand,

$$\begin{aligned}
d\eta \circ T(\nabla x) &:= (\langle \nabla \eta, T(\nabla x_1) \rangle, \dots, \langle \nabla \eta, T(\nabla x_m) \rangle) \\
&= (\langle \nabla \eta, T(\sum_{i=1}^n e_i(x_1) e_i) \rangle, \dots, \langle \nabla \eta, T(\sum_{i=1}^n e_i(x_m) e_i) \rangle) \\
&= \sum_{i=1}^n \langle \nabla \eta, T(e_i) \rangle (e_i(x_1), \dots, e_i(x_m)) \\
&= \sum_{i=1}^n \langle \nabla \eta, T(e_i) \rangle e_i(x) = \sum_{i=1}^n \langle T(\nabla \eta), e_i \rangle e_i = T(\nabla \eta).
\end{aligned} \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.16), we obtain

$$\operatorname{div}_\eta(T(\nabla x)) = \operatorname{tr}(\alpha(T(\cdot), \cdot)) + \operatorname{tr}(\nabla T) - T(\nabla \eta).$$

Now we compute

$$\begin{aligned}
\sum_{\ell=1}^m (\operatorname{div}_\eta(T(\nabla x_\ell)))^2 &= \|\operatorname{div}_\eta(T(\nabla x))\|^2 \\
&= \|\operatorname{tr}(\alpha(T(\cdot), \cdot))\|^2 + |\operatorname{tr}(\nabla T) - T(\nabla \eta)|^2
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\sum_{\ell=1}^m \operatorname{div}_\eta(T(\nabla x_\ell)) T(\nabla x_\ell, \nabla u_i) &= \sum_{\ell=1}^m \operatorname{div}_\eta(T(\nabla x_\ell)) T(\nabla u_i)(x_\ell) \\
&= \langle \operatorname{div}_\eta(T(\nabla x)), T(\nabla u_i) \rangle \\
&= \langle \operatorname{tr}(\nabla T) - T(\nabla \eta), T(\nabla u_i) \rangle \\
&= \langle \operatorname{tr}(\nabla T), T(\nabla u_i) \rangle - \langle T(\nabla \eta), T(\nabla u_i) \rangle.
\end{aligned} \tag{3.21}$$

Substituting (3.14), (3.15), (3.20) and (3.21) into (3.13) we complete the proof of the lemma. \square

3.1. Proof of Theorem 1.

Proof. We make use of Lemma 1. Firstly, we observe that

$$\begin{aligned} & \|\operatorname{tr}(\alpha(T(\cdot), \cdot))\|^2 + |\operatorname{tr}(\nabla T) - T(\nabla \eta)|^2 \\ &= \left\| \sum_{i=1}^n \alpha(T(e_i), e_i) \right\|^2 + |\operatorname{tr}(\nabla T)|^2 - 2\langle \operatorname{tr}(\nabla T), T(\nabla \eta) \rangle + |T(\nabla \eta)|^2. \end{aligned}$$

We have

$$\begin{aligned} |T(\nabla \eta)|^2 &= \sum_{i=1}^n \langle T(\nabla \eta), e_i \rangle^2 = \sum_{i=1}^n \langle \nabla \eta, T(e_i) \rangle^2 \\ &\leq |\nabla \eta|^2 \sum_{i=1}^n |T(e_i)|^2 = |\nabla \eta|^2 |T|^2, \end{aligned}$$

so, for $T_* = \sup_{\bar{\Omega}} |T|$ and $\eta_0 = \sup_{\bar{\Omega}} |\nabla \eta|$

$$|T(\nabla \eta)|^2 \leq T_*^2 \eta_0^2. \quad (3.22)$$

Further, for $T_0 = \sup_{\bar{\Omega}} |\operatorname{tr}(\nabla T)|$ we have

$$\begin{aligned} -2 \int_{\Omega} u_i^2 \langle \operatorname{tr}(\nabla T), T(\nabla \eta) \rangle \, \mathrm{d}m &\leq 2 \int_{\Omega} u_i^2 |\operatorname{tr}(\nabla T)| |T(\nabla \eta)| \, \mathrm{d}m \\ &\leq 2T_0 T_* \eta_0 \int_{\Omega} u_i^2 \, \mathrm{d}m \\ &\leq 2T_0 T_* \eta_0. \end{aligned} \quad (3.23)$$

We also have

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha(T(e_i), e_i) \right\|^2 &= \left\| \sum_{i=1}^n \sum_{k=n+1}^m \langle \alpha(T(e_i), e_i), e_k \rangle e_k \right\|^2 \\ &= \left\| \sum_{i=1}^n \sum_{k=n+1}^m \langle A_{e_k} e_i, T(e_i) \rangle e_k \right\|^2 \\ &= \left\| \sum_{k=n+1}^m \left(\sum_{i=1}^n \langle A_{e_k} e_i, T(e_i) \rangle \right) e_k \right\|^2 \\ &= \left\| \sum_{k=n+1}^m \langle A_{e_k}, T \rangle e_k \right\|^2. \end{aligned}$$

Estimating the right hand side we get

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha(T(e_i), e_i) \right\|^2 &\leq \sum_{k=n+1}^m |\langle A_{e_k}, T \rangle|^2 \sum_{k=n+1}^m |e_k|^2 \\ &\leq (m-n) \sum_{k=n+1}^m |A_{e_k}|^2 |T|^2 \\ &\leq (m-n)^2 A_0^2 T_*^2, \end{aligned} \quad (3.24)$$

where $A_0 = \max\{\sup_{\bar{\Omega}} |A_{e_k}|, k = n+1, \dots, m\}$, and each A_{e_k} is the Weingarten operator of the immersion in the direction e_k . We must notice that the tensor character of the Weingarten operator and the orientation of M allow us to assume that $\{e_k\}$, $k = n+1, \dots, m$ is a global orthonormal frame of normal vectors to M .

To complete the proof we need to estimate

$$\begin{aligned}
4 \int_{\Omega} u_i \langle \text{tr}(\nabla T), T(\nabla u_i) \rangle \text{dm} &\leq 4 \int_{\Omega} |u_i| |\text{tr}(\nabla T)| |T(\nabla u_i)| \text{dm} \\
&\leq 4 \left(\int_{\Omega} u_i^2 \text{dm} \right)^{\frac{1}{2}} \left(\int_{\Omega} |\text{tr}(\nabla T)|^2 |T(\nabla u_i)|^2 \text{dm} \right)^{\frac{1}{2}} \\
&\leq 4 T_0 \lambda_i^{\frac{1}{2}}
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
-4 \int_{\Omega} u_i \langle T(\nabla \eta), T(\nabla u_i) \rangle \text{dm} &\leq 4 \left(\int_{\Omega} |T(\nabla \eta)|^2 |T(\nabla u_i)|^2 \text{dm} \right)^{\frac{1}{2}} \\
&\leq 4 T_* \eta_0 \lambda_i^{\frac{1}{2}}.
\end{aligned} \tag{3.26}$$

Replacing (3.22), (3.23), (3.24), (3.25) and (3.26) into Lemma 1 we complete the proof of the theorem. \square

4. APPLICATIONS TO η -LAPLACIAN OPERATOR

In this section, we apply the results of the last section to obtain an upper bound for the eigenvalues from the problem (3.1) in the case $T = I$, that is, when the operator \mathcal{L} is the η -Laplacian.

4.1. Proof of Theorem 2.

Proof. From Lemma 1, for $T = I$ we can write

$$\begin{aligned}
n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 &\leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left[\int_{\Omega} u_i^2 (n^2 \|H\|^2 + |\nabla \eta|^2) \text{dm} \right. \\
&\quad \left. - 4 \int_{\Omega} u_i \langle \nabla \eta, \nabla u_i \rangle \text{dm} + 4 \lambda_i \right].
\end{aligned}$$

So, for $\eta_0 = \sup_{\bar{\Omega}} |\nabla \eta|$ and $H_0 = \sup_{\bar{\Omega}} \|H\|$ we have

$$n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (n^2 H_0^2 + \eta_0^2 + 4 \lambda_i - 4 \int_{\Omega} u_i \langle \nabla \eta, \nabla u_i \rangle \text{dm}). \tag{4.1}$$

But

$$\begin{aligned}
-4 \int_{\Omega} u_i \langle \nabla \eta, \nabla u_i \rangle \text{dm} &= -2 \int_{\Omega} \langle \nabla \eta, \nabla u_i^2 \rangle \text{dm} \\
&= 2 \int_{\Omega} u_i^2 \mathcal{L} \eta \text{dm} \\
&\leq 2 \bar{\eta}_0,
\end{aligned} \tag{4.2}$$

where $\bar{\eta}_0 = \sup_{\bar{\Omega}} \mathcal{L} \eta$. The conclusion of the theorem follows immediately by replacing (4.2) into (4.1). \square

We observe that in the case of the existence of a family of isometric immersions preserving the mean H , we get the same inequalities involving eigenvalues for all manifolds. For example, the family of isometric immersions that transforms catenoid in helicoid.

We present below some inequalities that are consequences of Theorem 2.

Corollary 1. *Under the assumption of Theorem 2 we have the following equations*

$$v_{k+1} \leq \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^k v_i \quad (4.3)$$

$$v_{k+1} \leq \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i + \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i \right)^2 - \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{j=1}^k \left(v_j - \frac{1}{k} \sum_{i=1}^k v_i \right)^2 \right]^{\frac{1}{2}} \quad (4.4)$$

$$v_{k+1} - v_k \leq 2 \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i \right)^2 - \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{j=1}^k \left(v_j - \frac{1}{k} \sum_{i=1}^k v_i \right)^2 \right]^{\frac{1}{2}} \quad (4.5)$$

Proof. From (1.16) we can write

$$\sum_{i=1}^k (v_{k+1} - v_i)(v_{k+1} - v_i) - \frac{4}{n} \sum_{i=1}^k (v_{k+1} - v_i)v_i \leq 0,$$

thus

$$\sum_{i=1}^k (v_{k+1} - v_i)v_{k+1} \leq \left(1 + \frac{4}{n}\right) \sum_{i=1}^k (v_{k+1} - v_i)v_i.$$

We affirm that

$$v_{k+1} \leq \left(1 + \frac{4}{n}\right) v_i, \quad (4.6)$$

for all $i = 1, \dots, k$. Indeed, otherwise there would be i_0 such that

$$v_{k+1} > \left(1 + \frac{4}{n}\right) v_{i_0} > \left(1 + \frac{4}{n}\right) v_{i_0-1} > \dots > \left(1 + \frac{4}{n}\right) v_1$$

which would imply

$$\sum_{i=1}^{i_0} (v_{i_0+1} - v_i)v_{i_0+1} > \left(1 + \frac{4}{n}\right) \sum_{i=1}^{i_0} (v_{i_0} - v_i)v_i$$

and would contradict inequality (1.16). Summing up the terms on the right hand side of (4.6) we derive

$$v_{k+1} \leq \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^k v_i$$

which proves (4.3). To prove (4.4), we note that (1.16) is equivalent to

$$\mathcal{P}(v_{k+1}) = k(v_{k+1})^2 - v_{k+1} \left(2 + \frac{4}{n}\right) \sum_{i=1}^k v_i + \left(1 + \frac{4}{n}\right) \sum_{i=1}^k (v_i)^2 \leq 0$$

then we can say that the discriminant of $\mathcal{P}(v_{k+1})$ satisfy

$$\mathcal{D} = \left(2 + \frac{4}{n}\right)^2 \left(\sum_{i=1}^k v_i\right)^2 - 4k \left(1 + \frac{4}{n}\right) \sum_{i=1}^k (v_i)^2 \geq 0. \quad (4.7)$$

Since $\mathcal{P}(v_{k+1}) \leq 0$ we have $r_{k+1}^\eta \leq v_{k+1} \leq R_{k+1}^\eta$, where r_{k+1}^η and R_{k+1}^η are the smaller and the biggest root of \mathcal{P} , respectively. Then

$$v_{k+1} \leq R_{k+1}^\eta = \frac{1}{2k} \left[\left(2 + \frac{4}{n}\right) \sum_{i=1}^k v_i + \sqrt{\mathcal{D}} \right]. \quad (4.8)$$

Substituting (4.7) into (4.8) we obtain

$$\begin{aligned} v_{k+1} &\leq \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i + \left[\left(\frac{1}{k} + \frac{2}{kn}\right)^2 \left(\sum_{i=1}^k v_i\right)^2 - \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^k (v_i)^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i + \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i\right)^2 + \frac{1}{k^2} \left(1 + \frac{4}{n}\right) \left(\sum_{i=1}^k v_i\right)^2 \right. \\ &\quad \left. - \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^k (v_i)^2 \right]^{\frac{1}{2}} \end{aligned}$$

or equivalently

$$\begin{aligned} v_{k+1} &\leq \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i + \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i\right)^2 - \frac{1}{k} \left(1 + \frac{4}{n}\right) \left(\sum_{i=1}^k (v_i)^2 - \frac{1}{k} \left(\sum_{i=1}^k v_i\right)^2\right) \right]^{\frac{1}{2}} \\ &= \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i + \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i\right)^2 \right. \\ &\quad \left. - \frac{1}{k} \left(1 + \frac{4}{n}\right) \left(\sum_{i=1}^k (v_i)^2 - \frac{2}{k} \left(\sum_{i=1}^k v_i\right)^2 + \frac{1}{k} \left(\sum_{i=1}^k v_i\right)^2\right) \right]^{\frac{1}{2}} \\ &= \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i + \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i\right)^2 \right. \\ &\quad \left. - \frac{1}{k} \left(1 + \frac{4}{n}\right) \left(\sum_{i=1}^k (v_i)^2 - \frac{2}{k} \sum_{i,j=1}^k v_i v_j + \frac{1}{k} \left(\sum_{i=1}^k v_i\right)^2\right) \right]^{\frac{1}{2}}, \end{aligned}$$

so

$$v_{k+1} \leq \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i + \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i\right)^2 - \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{j=1}^k \left(v_j - \frac{1}{k} \sum_{i=1}^k v_i\right)^2 \right]^{\frac{1}{2}}.$$

Finally, as (1.16) is true for all k , it follows that

$$\sum_{i=1}^k (v_k - v_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (v_k - v_i) v_i,$$

that is, we can observe again that the polynomial $\mathcal{P}(v_k) \leq 0$. Analogously we have

$$v_k \geq r_k^\eta = \frac{1}{k} \left(1 + \frac{2}{n}\right) \sum_{i=1}^k v_i - \left[\left(\frac{2}{kn} \sum_{i=1}^k v_i\right)^2 - \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{j=1}^k \left(v_j - \frac{1}{k} \sum_{i=1}^k v_i\right)^2 \right]^{\frac{1}{2}}. \quad (4.9)$$

Hence from (4.4) and (4.9) we obtain (4.5). \square

We note that (4.3) is a second-Yang type inequality and (4.5) generalizes (1.6). We also obtain the following lower bound for sums of the first k eigenvalues, which extends inequality (1.11).

Corollary 2. *Under the assumption of Theorem 2 we have*

$$\frac{1}{k} \sum_{i=1}^k v_i \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}} \quad \text{for } k \geq 1.$$

Proof. Using inequality (1.16) and Lemma 2 we obtain

$$F_{k+1} \leq C(n, k) \left(\frac{k+1}{k} \right)^{\frac{4}{n}} F_k \leq \left(\frac{k+1}{k} \right)^{\frac{4}{n}} F_k,$$

that is,

$$\frac{F_{k+1}}{(k+1)^{\frac{4}{n}}} \leq \frac{F_k}{k^{\frac{4}{n}}}.$$

More generally,

$$\frac{F_{k+l}}{(k+l)^{\frac{4}{n}}} \leq \frac{F_k}{k^{\frac{4}{n}}}, \quad (4.10)$$

for any positive integer l . Furthermore, by Lemma 2 we have

$$\begin{aligned} \frac{F_k}{k^{\frac{4}{n}}} &= \frac{(1 + \frac{2}{n}) \Lambda_k^2 - T_k}{k^{\frac{4}{n}}} = \frac{\frac{2}{n} \left(\frac{1}{k} \sum_{i=1}^k v_i \right)^2 - \frac{1}{k} \sum_{i=1}^k (v_i - \frac{1}{k} \sum_{i=1}^k v_i)}{k^{\frac{4}{n}}} \\ &\leq \frac{\frac{2}{n} \left(\frac{1}{k} \sum_{i=1}^k v_i \right)^2}{k^{\frac{4}{n}}} \end{aligned} \quad (4.11)$$

and

$$\frac{F_{k+l}}{(k+l)^{\frac{4}{n}}} = \left(1 + \frac{2}{n} \right) \left(\frac{\frac{1}{k+l} \sum_{i=1}^{k+l} v_i}{(k+l)^{\frac{2}{n}}} \right)^2 - \frac{\frac{1}{k+l} \sum_{i=1}^{k+l} v_i^2}{(k+l)^{\frac{4}{n}}}. \quad (4.12)$$

On the other hand, the Weyl's asymptotic formula says that

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k \lambda_i}{k^{\frac{2}{n}}} = \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} \quad (4.13)$$

and

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k \lambda_i^2}{k^{\frac{4}{n}}} = \frac{n}{n+4} \frac{16\pi^4}{(\omega_n \text{vol} \Omega)^{\frac{4}{n}}}, \quad (4.14)$$

then (4.13) implies

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k \lambda_i}{k^{\frac{2}{n}}} &= \lim_{k \rightarrow \infty} \frac{\left(\frac{1}{k} \sum_{i=1}^k \lambda_i \right) + \frac{n^2 H_0^2 + \eta_0^2 + 2\eta_0}{4}}{k^{\frac{2}{n}}} \\ &= \lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k (\lambda_i + \frac{n^2 H_0^2 + \eta_0^2 + 2\eta_0}{4})}{k^{\frac{2}{n}}}. \end{aligned}$$

Therefore

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k v_i}{k^{\frac{2}{n}}} = \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}}. \quad (4.15)$$

Analogously, from (4.14) we get

$$\begin{aligned} &\lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k \left(\lambda_i + \frac{n^2 H_0^2 + \phi_0^2 + 4\phi_0}{4(1+\phi_0)} \right)^2}{k^{\frac{4}{n}}} \\ &= \frac{n}{n+4} \frac{16\pi^4}{(\omega_n \text{vol} \Omega)^{\frac{4}{n}}} + \lim_{k \rightarrow \infty} \frac{\left(\frac{n^2 H_0^2 + \phi_0^2 + 4\phi_0}{4(1+\phi_0)} \right)^2}{k^{\frac{2}{n}}} + \lim_{k \rightarrow \infty} \frac{2 \frac{n^2 H_0^2 + \phi_0^2 + 4\phi_0}{4(1+\phi_0)} \frac{1}{k} \sum_{i=1}^k \lambda_i}{k^{\frac{2}{n}}}. \end{aligned}$$

Hence

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k v_i^2}{k^{\frac{4}{n}}} = \frac{n}{n+4} \frac{16\pi^4}{(\omega_n \text{vol} \Omega)^{\frac{4}{n}}}. \quad (4.16)$$

Thus, from (4.12), (4.15) and (4.16) we obtain

$$\begin{aligned} \lim_{l \rightarrow \infty} \frac{F_{k+l}}{(k+l)^{\frac{4}{n}}} &= \lim_{l \rightarrow \infty} \left(1 + \frac{2}{n}\right) \left(\frac{\frac{1}{k+l} \sum_{i=1}^{k+l} v_i}{(k+l)^{\frac{2}{n}}}\right)^2 - \lim_{l \rightarrow \infty} \frac{\frac{1}{k+l} \sum_{i=1}^{k+l} v_i^2}{(k+l)^{\frac{4}{n}}} \\ &= \frac{2n}{(n+2)(n+4)} \frac{16\pi^4}{(\omega_n \text{vol} \Omega)^{\frac{4}{n}}}. \end{aligned} \quad (4.17)$$

Finally, from (4.10), (4.11) and (4.17) we conclude, for any positive integer k ,

$$\frac{\frac{2}{n} \left(\frac{1}{k} \sum_{i=1}^k v_i\right)^2}{k^{\frac{4}{n}}} \geq \frac{2n}{(n+2)(n+4)} \frac{16\pi^4}{(\omega_n \text{vol} \Omega)^{\frac{4}{n}}}$$

that is,

$$\frac{1}{k} \sum_{i=1}^k v_i \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

□

The next result extends the estimate (1.7) for the η -Laplacian operator. It is immediate from (1.16) and Lemma 3.

Corollary 3. *Under the assumption of Theorem 2 we have*

$$v_{k+1} \leq \left(1 + \frac{4}{n}\right) k^{\frac{2}{n}} v_1.$$

We point out that the previous inequality agrees with the Weyl's asymptotic formula.

5. APPENDIX

In this appendix we will present the technical results used above.

Lemma 2. *Let $\eta_1 \leq \eta_2 \leq \dots \leq \eta_{k+1}$ be positive real numbers satisfying*

$$\sum_{i=1}^k (\eta_{k+1} - \eta_i)^2 \leq \frac{4c}{n} \sum_{i=1}^k (\eta_{k+1} - \eta_i) \eta_i,$$

for c, n positive real numbers, then

$$F_{k+1} \leq C(n, k, c) \left(\frac{k+1}{k}\right)^{\frac{4c}{n}} F_k,$$

where

$$F_k := \left(1 + \frac{2c}{n}\right) \Lambda_k^2 - T_k, \quad \Lambda_k := \frac{1}{k} \sum_{i=1}^k \eta_i, \quad T_k := \frac{1}{k} \sum_{i=1}^k \eta_i^2,$$

and

$$0 < C(n, k, c) = 1 - \frac{c}{3n} \left(\frac{k}{k+1}\right)^{\frac{4c}{n}} \frac{\left(1 + \frac{2c}{n}\right) \left(1 + \frac{4c}{n}\right)}{(k+1)^3} < 1.$$

Lemma 3. *Let $\eta_1 \leq \eta_2 \leq \dots \leq \eta_{k+1}$ be positive real numbers satisfying*

$$\sum_{i=1}^k (\eta_{k+1} - \eta_i)^2 \leq \frac{4c}{n} \sum_{i=1}^k (\eta_{k+1} - \eta_i) \eta_i,$$

for c, n positive real numbers, then

$$\eta_{k+1} \leq \left(1 + \frac{4c}{n}\right) k^{\frac{2c}{n}} \eta_1.$$

Observe that the two lemmas above were already obtained by Cheng and Yang [7] in the case of $c = 1$. Notice that the same technique works for arbitrary c .

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